3.9 Fluid dynamics

Ideal fluids[a]

Continuity[b]	$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\boldsymbol{v})=0$$	(3.285)	ρ	density
			$\boldsymbol{v}$	fluid velocity field
			t	time
Kelvin circulation	$$\Gamma=\oint\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{l}=\text{constant}$$	(3.286)	Γ	circulation
			$\mathrm{d}\boldsymbol{l}$	loop element
	$$=\int_S\boldsymbol{\omega}\cdot\mathrm{d}\boldsymbol{s}$$	(3.287)	$\mathrm{d}\boldsymbol{s}$	element of surface bounded by loop
			$\boldsymbol{\omega}$	vorticity ($=\nabla\times\boldsymbol{v}$)
Euler's equation[c]	$$\frac{\partial\boldsymbol{v}}{\partial t}+(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}=-\frac{\nabla p}{\rho}+\boldsymbol{g}$$	(3.288)	p	pressure
			$\boldsymbol{g}$	gravitational field strength
	or $$\frac{\partial}{\partial t}(\nabla\times\boldsymbol{v})=\nabla\times[\boldsymbol{v}\times(\nabla\times\boldsymbol{v})]$$	(3.289)	$(\boldsymbol{v}\cdot\nabla)$	advective operator
Bernoulli's equation (incompressible flow)	$$\frac{1}{2}\rho v^2+p+\rho gz=\text{constant}$$	(3.290)	z	altitude
Bernoulli's equation (compressible adiabatic flow)[d]	$$\frac{1}{2}v^2+\frac{\gamma}{\gamma-1}\frac{p}{\rho}+gz=\text{constant}$$	(3.291)	γ	ratio of specific heat capacities (c_p/c_V)
	$$=\frac{1}{2}v^2+c_pT+gz$$	(3.292)	c_p	specific heat capacity at constant pressure
			T	temperature
Hydrostatics	$$\nabla p=\rho\boldsymbol{g}$$	(3.293)		
Adiabatic lapse rate (ideal gas)	$$\frac{\mathrm{d}T}{\mathrm{d}z}=-\frac{g}{c_p}$$	(3.294)		

[a]No thermal conductivity or viscosity.
[b]True generally.
[c]The second form of Euler's equation applies to incompressible flow only.
[d]Equation (3.292) is true only for an ideal gas.

Potential flow[a]

Velocity potential	$$\boldsymbol{v}=\nabla\phi$$	(3.295)	$\boldsymbol{v}$	velocity
	$$\nabla^2\phi=0$$	(3.296)	ϕ	velocity potential
Vorticity condition	$$\boldsymbol{\omega}=\nabla\times\boldsymbol{v}=0$$	(3.297)	$\boldsymbol{\omega}$	vorticity
			$\boldsymbol{F}$	drag force on moving sphere
Drag force on a sphere[b]	$$\boldsymbol{F}=-\frac{2}{3}\pi\rho a^3\dot{\boldsymbol{u}}=-\frac{1}{2}M_{\mathrm{d}}\dot{\boldsymbol{u}}$$	(3.298)	a	sphere radius
			$\dot{\boldsymbol{u}}$	sphere acceleration
			ρ	fluid density
			M_{d}	displaced fluid mass

[a]For incompressible fluids.
[b]The effect of this drag force is to give the sphere an additional effective mass equal to half the mass of fluid displaced.

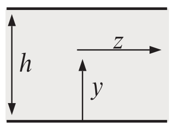
Viscous flow (incompressible)[a]

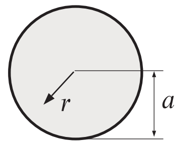
Fluid stress	$\tau_{ij} = -p\delta_{ij} + \eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$	(3.299)	τ_{ij} fluid stress tensor p hydrostatic pressure η shear viscosity v_i velocity along i axis δ_{ij} Kronecker delta
Navier–Stokes equation[b]	$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\frac{\nabla p}{\rho} - \frac{\eta}{\rho}\nabla\times\boldsymbol{\omega} + \boldsymbol{g}$	(3.300)	$\boldsymbol{v}$ fluid velocity field $\boldsymbol{\omega}$ vorticity $\boldsymbol{g}$ gravitational acceleration ρ density
	$= -\frac{\nabla p}{\rho} + \frac{\eta}{\rho}\nabla^2\boldsymbol{v} + \boldsymbol{g}$	(3.301)	
Kinematic viscosity	$\nu = \eta/\rho$	(3.302)	ν kinematic viscosity

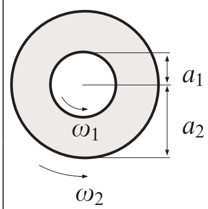
[a]I.e., $\nabla\cdot\boldsymbol{v} = 0$, $\eta \neq 0$.
[b]Neglecting bulk (second) viscosity.

Laminar viscous flow

Between parallel plates	$v_z(y) = \frac{1}{2\eta}y(h-y)\frac{\partial p}{\partial z}$	(3.303)	v_z flow velocity z direction of flow y distance from plate η shear viscosity p pressure
Along a circular pipe[a]	$v_z(r) = \frac{1}{4\eta}(a^2 - r^2)\frac{\partial p}{\partial z}$	(3.304)	r distance from pipe axis a pipe radius V volume
	$Q = \frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\pi a^4}{8\eta}\frac{\partial p}{\partial z}$	(3.305)	
Circulating between concentric rotating cylinders[b]	$G_z = \frac{4\pi\eta a_1^2 a_2^2}{a_2^2 - a_1^2}(\omega_2 - \omega_1)$	(3.306)	G_z axial couple between cylinders per unit length ω_i angular velocity of ith cylinder
Along an annular pipe	$Q = \frac{\pi}{8\eta}\frac{\partial p}{\partial z}\left[a_2^4 - a_1^4 - \frac{(a_2^2 - a_1^2)^2}{\ln(a_2/a_1)}\right]$	(3.307)	a_1 inner radius a_2 outer radius Q volume discharge rate







[a]Poiseuille flow.
[b]Couette flow.

Drag[a]

On a sphere (Stokes's law)	$F = 6\pi a\eta v$	(3.308)	F drag force a radius
On a disk, broadside to flow	$F = 16 a\eta v$	(3.309)	v velocity η shear viscosity
On a disk, edge on to flow	$F = 32 a\eta v/3$	(3.310)	

[a]For Reynolds numbers $\ll 1$.

Characteristic numbers

Reynolds number	$\mathsf{Re} = \frac{\rho U L}{\eta} = \frac{\text{inertial force}}{\text{viscous force}}$	(3.311)	Re ρ U L η	Reynolds number density characteristic velocity characteristic scale-length shear viscosity
Froude number[a]	$\mathsf{F} = \frac{U^2}{Lg} = \frac{\text{inertial force}}{\text{gravitational force}}$	(3.312)	F g	Froude number gravitational acceleration
Strouhal number[b]	$\mathsf{S} = \frac{U\tau}{L} = \frac{\text{evolution scale}}{\text{physical scale}}$	(3.313)	S τ	Strouhal number characteristic timescale
Prandtl number	$\mathsf{P} = \frac{\eta c_p}{\lambda} = \frac{\text{momentum transport}}{\text{heat transport}}$	(3.314)	P c_p λ	Prandtl number Specific heat capacity at constant pressure thermal conductivity
Mach number	$\mathsf{M} = \frac{U}{c} = \frac{\text{speed}}{\text{sound speed}}$	(3.315)	M c	Mach number sound speed
Rossby number	$\mathsf{Ro} = \frac{U}{\Omega L} = \frac{\text{inertial force}}{\text{Coriolis force}}$	(3.316)	Ro Ω	Rossby number angular velocity

[a]Sometimes the square root of this expression. L is usually the fluid depth.
[b]Sometimes the reciprocal of this expression.

Fluid waves

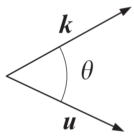
Sound waves	$v_p = \left(\frac{K}{\rho}\right)^{1/2} = \left(\frac{\mathrm{d}p}{\mathrm{d}\rho}\right)^{1/2}$	(3.317)	v_p K p ρ	wave (phase) speed bulk modulus pressure density
In an ideal gas (adiabatic conditions)[a]	$v_p = \left(\frac{\gamma RT}{\mu}\right)^{1/2} = \left(\frac{\gamma p}{\rho}\right)^{1/2}$	(3.318)	γ R T μ	ratio of heat capacities molar gas constant (absolute) temperature mean molecular mass
Gravity waves on a liquid surface[b]	$\omega^2 = gk \tanh kh$	(3.319)	v_g h λ k g ω	group speed of wave liquid depth wavelength wavenumber gravitational acceleration angular frequency
	$v_g \simeq \begin{cases} \frac{1}{2}\left(\frac{g}{k}\right)^{1/2} & (h \gg \lambda) \\ (gh)^{1/2} & (h \ll \lambda) \end{cases}$	(3.320)		
Capillary waves (ripples)[c]	$\omega^2 = \frac{\sigma k^3}{\rho}$	(3.321)	σ	surface tension
Capillary–gravity waves ($h \gg \lambda$)	$\omega^2 = gk + \frac{\sigma k^3}{\rho}$	(3.322)		

[a]If the waves are isothermal rather than adiabatic then $v_p = (p/\rho)^{1/2}$.
[b]Amplitude $\ll$ wavelength.
[c]In the limit $k^2 \gg g\rho/\sigma$.

Doppler effect[a]

Source at rest, observer moving at u	$\frac{\nu'}{\nu}=1-\frac{\lvert\boldsymbol{u}\rvert}{v_{\rm p}}\cos\theta$	(3.323)
Observer at rest, source moving at u	$\frac{\nu''}{\nu}=\frac{1}{1-\frac{\lvert\boldsymbol{u}\rvert}{v_{\rm p}}\cos\theta}$	(3.324)

ν', ν''	observed frequency
ν	emitted frequency
$v_{\rm p}$	wave (phase) speed in fluid
$\boldsymbol{u}$	velocity
θ	angle between wavevector, $\boldsymbol{k}$, and $\boldsymbol{u}$



[a]For plane waves in a stationary fluid.

Wave speeds

Phase speed	$v_{\rm p}=\frac{\omega}{k}=\nu\lambda$	(3.325)
Group speed	$v_{\rm g}=\frac{{\rm d}\omega}{{\rm d}k}$	(3.326)
	$=v_{\rm p}-\lambda\frac{{\rm d}v_{\rm p}}{{\rm d}\lambda}$	(3.327)

$v_{\rm p}$	phase speed
ν	frequency
ω	angular frequency ($=2\pi\nu$)
λ	wavelength
k	wavenumber ($=2\pi/\lambda$)
$v_{\rm g}$	group speed

Shocks

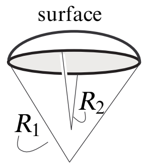
Mach wedge[a]	$\sin\theta_{\rm w}=\frac{v_{\rm p}}{v_{\rm b}}$	(3.328)
Kelvin wedge[b]	$\lambda_{\rm K}=\frac{4\pi v_{\rm b}^2}{3g}$	(3.329)
	$\theta_{\rm w}=\arcsin(1/3)=19^\circ.5$	(3.330)
Spherical adiabatic shock[c]	$r\simeq\left(\frac{Et^2}{\rho_0}\right)^{1/5}$	(3.331)
Rankine–Hugoniot shock relations[d]	$\frac{p_2}{p_1}=\frac{2\gamma\mathrm{M}_1^2-(\gamma-1)}{\gamma+1}$	(3.332)
	$\frac{v_1}{v_2}=\frac{\rho_2}{\rho_1}=\frac{\gamma+1}{(\gamma-1)+2/\mathrm{M}_1^2}$	(3.333)
	$\frac{T_2}{T_1}=\frac{[2\gamma\mathrm{M}_1^2-(\gamma-1)][2+(\gamma-1)\mathrm{M}_1^2]}{(\gamma+1)^2\mathrm{M}_1^2}$	(3.334)

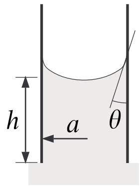
$\theta_{\rm w}$	wedge semi-angle
$v_{\rm p}$	wave (phase) speed
$v_{\rm b}$	body speed
$\lambda_{\rm K}$	characteristic wavelength
g	gravitational acceleration
r	shock radius
E	energy release
t	time
ρ_0	density of undisturbed medium
1	upstream values
2	downstream values
p	pressure
v	velocity
T	temperature
ρ	density
γ	ratio of specific heats
M	Mach number

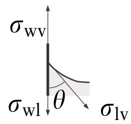
[a]Approximating the wake generated by supersonic motion of a body in a nondispersive medium.
[b]For gravity waves, e.g., in the wake of a boat. Note that the wedge semi-angle is independent of $v_{\rm b}$.
[c]Sedov–Taylor relation.
[d]Solutions for a steady, normal shock, in the frame moving with the shock front. If $\gamma=5/3$ then $v_1/v_2\leq 4$.

Surface tension

Definition	$\sigma_{\mathrm{lv}} = \dfrac{\text{surface energy}}{\text{area}}$	(3.335)	σ_{lv} surface tension (liquid/vapour interface)
	$= \dfrac{\text{surface tension}}{\text{length}}$	(3.336)	
Laplace's formula[a]	$\Delta p = \sigma_{\mathrm{lv}} \left(\dfrac{1}{R_1} + \dfrac{1}{R_2} \right)$	(3.337)	Δp pressure difference over surface; R_i principal radii of curvature
Capillary constant	$c_{\mathrm{c}} = \left(\dfrac{2\sigma_{\mathrm{lv}}}{g\rho} \right)^{1/2}$	(3.338)	c_{c} capillary constant; ρ liquid density; g gravitational acceleration
Capillary rise (circular tube)	$h = \dfrac{2\sigma_{\mathrm{lv}} \cos\theta}{\rho g a}$	(3.339)	h rise height; θ contact angle; a tube radius
Contact angle	$\cos\theta = \dfrac{\sigma_{\mathrm{wv}} - \sigma_{\mathrm{wl}}}{\sigma_{\mathrm{lv}}}$	(3.340)	σ_{wv} wall/vapour surface tension; σ_{wl} wall/liquid surface tension







[a]For a spherical bubble in a liquid $\Delta p = 2\sigma_{\mathrm{lv}}/R$. For a soap bubble (two surfaces) $\Delta p = 4\sigma_{\mathrm{lv}}/R$.